Journal of Computational and Applied Mathematics 117 (2000) 121–135

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Weighted rational cubic spline interpolation and its application

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Received 16 January 1998; received in revised form 24 July 1999

Abstract

In Qi Duan et al. (Korean J. Comput. Appl. Math. 6 (1) (1999) 203–215), the authors have discussed constrained interpolation problems by means of rational cubic spline interpolation with linear denominators, but there are still some cases in which the constrained interpolation cannot be solved. In this paper, the weighted rational cubic spline interpolation has been constructed using the rational cubic spline with linear denominator and the rational cubic spline based on function values. By these, the problems to constrain the weighted rational interpolation curves to lie strictly above or below a given piecewise linear curve and between two given piecewise linear curves can be solved completely. Also, the approximation properties of these weighted rational cubic splines are studied. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Rational spline; Cubic spline; Constrained interpolation; Weighted rational interpolation; Approximation

1. Introduction

Spline interpolation is a useful and powerful tool for curve and surface design. Many authors have studied several kinds of splines [1,2,6,9–12] for curve and surface control. In recent years, the rational spline, especially the rational cubic spline, and its application to shape control have been considered [3,7,8,13–15]. In [4], the authors have studied some approximation properties of some typical rational cubic splines, such as the rational cubic spline with cubic denominator [13], the rational cubic spline with quadratic denominator [8], and the rational cubic spline with linear denominator [5]. From these studies, it is found that the rational cubic spline with linear denominator gives a better approximation to the function being interpolated [4]. Moreover, it is found that it can be easily used to control the shape of a curve just by choosing suitable parameters, such as constraining

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PII: S0377-0427(99)00336-2

the curve to lie below (or above) a straight line or quadratic curve [5]. There are, however, still some cases for which there are no parameters that can be used in order to control the shape of the curve. To overcome this problem, a weighted rational cubic spline interpolation is constructed in this paper.

The paper is structured as follows. In Section 2, the C^1-continuous weighted rational cubic spline is derived by taking a linear combination of the rational cubic spline with linear denominator [5] and the rational cubic spline based on function values [3]. Under certain conditions, a C^2-continuous weighted rational cubic spline is also obtained. Section 3 deals with the problem that constrains the weighted rational interpolant curves to lie above or below a given piecewise linear curve and between two given piecewise linear curves. A numerical example is given to illustrate these phenomena. In Section 4, the approximation properties of the weighted rational spline interpolation are studied.

2. Weighted rational cubic spline interpolation

Given a data set $\{(t_i, f_i, d_i),\ i = 0, 1, \ldots, n, n+1\}$, f_i and d_i are the function value and the first derivative value defined at the knot, respectively, where $t_0 < t_1 < \cdots < t_n < t_{n+1}$ are the knots. Let $h_i = t_{i+1} - t_i$, $\theta = (t - t_i)/h_i$, and let α_i and β_i be positive parameters. There is a C^1-continuous, piecewise, rational cubic spline with linear denominator [5] defined in the interpolatory interval $[t_0, t_{n+1}]$ as follows:

$$P^*(t)\big|_{[t_i,t_{i+1}]} = \frac{p_i^*(t)}{q_i^*(t)}, \quad i = 0, 1, \ldots, n, \tag{1}$$

where

$$p_i^*(t) = (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i^* + \theta^2(1-\theta)W_i^* + \theta^3\beta_i f_{i+1},$$

$$q_i^*(t) = (1-\theta)\alpha_i + \theta\beta_i$$

and

$$V_i^* = (2\alpha_i + \beta_i)f_i + \alpha_i h_i d_i,$$

$$W_i^* = (\alpha_i + 2\beta_i)f_{i+1} - \beta_i h_i d_{i+1}.$$

This rational cubic spline satisfies

$$P^*(t_i) = f_i, \qquad P^{*\prime}(t_i) = d_i, \qquad i = 0, 1, \ldots, n, n+1.$$

Weighted methods are used frequently in numerical analysis, but appear not very often in interpolation. In order to use the weighted method here, the C^1-continuous, piecewise, rational cubic spline interpolation based on function values [3] will be restated as follows:

$$P_*(t)\big|_{[t_i,t_{i+1}]} = \frac{p_{i,*}(t)}{q_{i,*}(t)}, \quad i = 0, 1, \ldots, n-1, \tag{2}$$

where

$$p_{i,*}(t) = (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_{i,*} + \theta^2(1-\theta)W_{i,*} + \theta^3\beta_i f_{i+1},$$

$$q_{i,*}(t) = (1-\theta)\alpha_i + \theta\beta_i$$

and

$$V_{i,*} = (\alpha_i + \beta_i)f_i + \alpha_i f_{i+1},$$

$$W_{i,*} = (\alpha_i + 2\beta_i)f_{i+1} - h_i\beta_i\Delta_{i+1},$$

in which $\Delta_i = (f_{i+1} - f_i)/h_i$. This rational cubic spline, $P_*(t)$, satisfies

$$P_*(t_i) = f_i, \quad P_*'(t_i) = \Delta_i, \quad i = 0, 1, \dots, n.$$

Now, the weighted rational cubic spline can be constructed by using the rational cubic splines defined by (1) and (2) as follows. Let

$$\begin{aligned} P(t) &= \lambda P^*(t) + (1-\lambda)P_*(t) \\ &= \frac{p_i(t)}{q_i(t)}, \quad t \in [t_i, t_{i+1}], \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{3}$$

where

$$p_i(t) = (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1},$$

$$q_i(t) = (1-\theta)\alpha_i + \theta\beta_i$$

and

$$V_i = (\lambda\alpha_i + \alpha_i + \beta_i)f_i + (1-\lambda)\alpha_i f_{i+1} + \lambda\alpha_i h_i d_i,$$

$$W_i = (\alpha_i + 2\beta_i)f_{i+1} - (1-\lambda)h_i\beta_i\Delta_{i+1} - \lambda\beta_i h_i d_{i+1}$$

with $\lambda \in \mathbb{R}^+$. It is obvious that $P(t)$, which is obtained as a linear combination of the interpolations $P^*(t)$ and $P_*(t)$, is a C^1-continuous, piecewise, rational cubic spline (with linear denominator). Note that $P^*(t)$ and $P_*(t)$ are defined in $[t_0, t_{n+1}]$ and $[t_0, t_n]$, respectively: thus, $P(t)$ is defined in $[t_0, t_n]$ and satisfies

$$P(t_i) = f_i, \quad P'(t_i) = \lambda d_i + (1-\lambda)\Delta_i, \quad i = 0, 1, \dots, n.$$

Furthermore, for a real number λ, the C^2-continuous rational cubic spline defined by (3) can be constructed by requiring

$$P''(t_i+) = P''(t_i-)$$

for $i = 1, 2, \dots, n-1$. These conditions lead to the following tri-diagonal system of linear equations:

$$\begin{aligned} &\lambda h_i \frac{\alpha_{i-1}}{\beta_{i-1}} d_{i-1} + \lambda\left(h_i\left(1 + \frac{\alpha_{i-1}}{\beta_{i-1}}\right) + h_{i-1}\left(1 + \frac{\beta_i}{\alpha_i}\right)\right) d_i + \lambda h_{i-1}\frac{\beta_i}{\alpha_i} d_{i+1} \\ &\quad = h_i\left(1 + (\lambda+1)\frac{\alpha_{i-1}}{\beta_{i-1}}\right)\Delta_{i-1} + \left(h_i(\lambda-1)\left(1 + \frac{\alpha_{i-1}}{\beta_{i-1}}\right) + h_{i-1}\left(\lambda + (\lambda+1)\frac{\beta_i}{\alpha_i}\right)\right)\Delta_i \\ &\qquad + h_{i-1}(\lambda-1)\frac{\beta_i}{\alpha_i}\Delta_{i+1}, \quad i = 1, 2, \dots, n-1. \end{aligned} \tag{4}$$

If λ, α_i, β_i, $i = 0, 1, \dots, n-1$, and the auxiliary end conditions are given, then the tri-diagonal system (4) can be easily solved for d_i. Since the matrix in Eq. (4) is diagonally dominant, the solution exists.

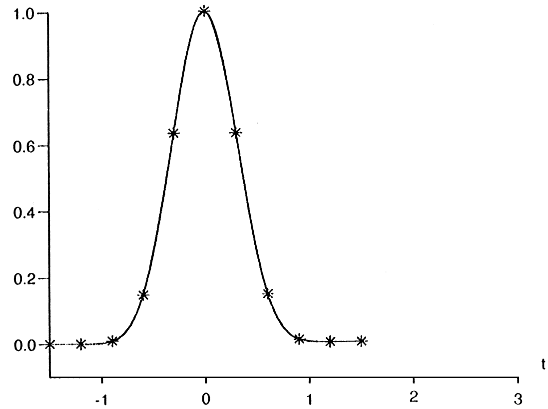

Fig. 1. The C^1-continuous, rational cubic spline [Eq. (3)] with $\alpha_i = 1$, $\beta_i = 2$ and $\lambda = 1$, and the original function $f(t) = \cos^{10}(t)$.

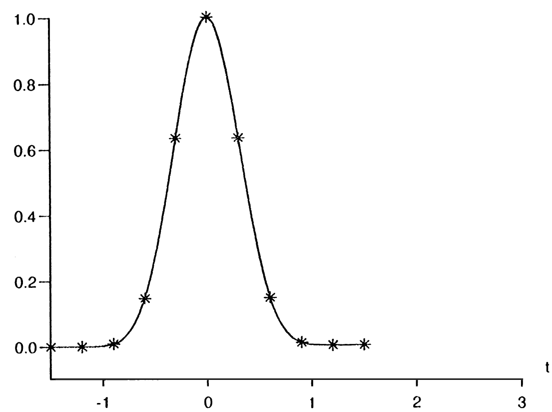

Fig. 2. The C^1-continuous, rational cubic spline [Eq. (3)] with $\alpha_i = 1$, $\beta_i = 2$ and $\lambda = 1.1$, and the original function $f(t) = \cos^{10}(t)$.

Some examples of rational cubic spline interpolation with linear denominator are given in Figs. 1–6. Figs. 1–3 show the C^1-continuous, rational cubic spline [Eq. (3)] (with $d_i = f_i'$, $\alpha_i = 1$ and $\beta_i = 2$, and for different values of λ) that fits the function $f(t) = \cos^{10}(t)$ at equally spaced knots $t_i = -1.5 + ih$, $i = 0, 1, \ldots, 10$ and $h = 0.3$. It can be seen from Figs. 1 and 2 (with $\lambda = 1$ and 1.1, respectively) that the C^1-continuous, rational cubic spline [Eq. (3)] is very similar to the original function $f(t) = \cos^{10}(t)$, whereas in Fig. 3 ($\lambda = 3$) the difference between the rational cubic spline [Eq. (3)] and the original function is quite large. These results are expected from the error constants $c_i(\eta_i, \lambda)$ (see Section 4, Fig. 9).

Figs. 4 and 5 show the C^2-continuous, rational cubic spline [Eqs. (3) and (4) (solved for d_i) for $\alpha_i = 1$, $\beta_i = 2$, and $\lambda = 1$ and 3, respectively], with natural end conditions ($d_0 = d_n = 0$), together with the original function and the same interpolation points as before. From these figures, it can be seen that the results obtained by the C^2-continuous, rational cubic spline interpolation are very similar to the original function.

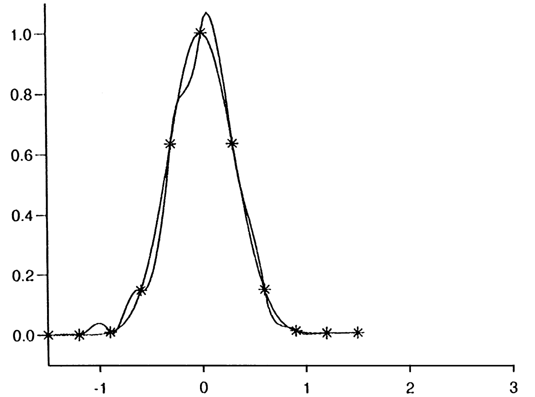

Fig. 3. The C^1-continuous, rational cubic spline [Eq. (3)] with $\alpha_i = 1$, $\beta_i = 2$ and $\lambda = 3$, and the original function $f(t) = \cos^{10}(t)$.

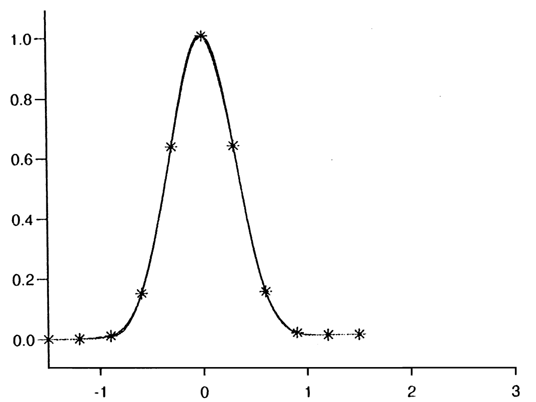

Fig. 4. The C^2-continuous, rational cubic spline [Eqs. (3) and (7)] with $\alpha_i = 1$, $\beta_i = 2$ and $\lambda = 1$, and the original function $f(t) = \cos^{10}(t)$.

Finally, Fig. 6 shows the C^2-continuous, rational cubic spline [Eqs. (3) and (4)] with $\alpha_i = 1$, $\beta_i = 2$ and $\lambda = 1$, together with the original curves ($f(t)^2 - t = 0$ and $f(t)^2 + 4f(t) + t + 2 = 0$) and some interpolation points. The C^2-continuous, rational cubic spline is obtained from the parametric form $(x(t), y(t))$, using all the data points in one run. The parameterization adopted in this case is the unit parameterization ($t_i = i$).

3. Constrained interpolation and the existence conditions

Given a function $g(t)$ and a data set $\{(t_i, f_i, d_i),\ i = 0, 1, \ldots, n\}$ with

$$f_i \geqslant g(t_i) \quad \text{or } f_i \leqslant g(t_i) \quad i = 0, 1, \ldots, n,$$

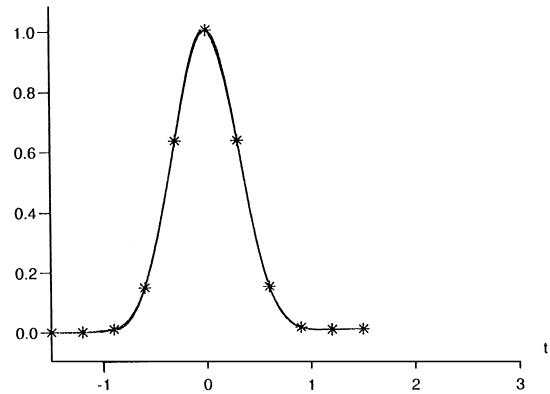

Fig. 5. The C^2-continuous, rational cubic spline [Eqs. (3) and (7)] with $\alpha_i = 1$, $\beta_i = 2$ and $\lambda = 3$, and the original function $f(t) = \cos^{10}(t)$.

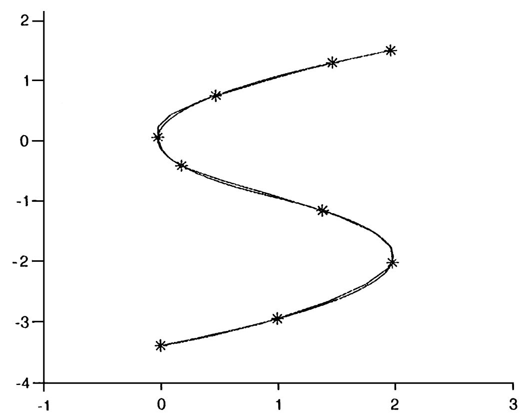

Fig. 6. The C^2-continuous, rational cubic spline [Eqs. (3) and (6)] with $\alpha_i = 1$, $\beta_i = 2$ and $\lambda = 1$, and the original curves $(f(t)^2 - t = 0,\ f(t)^2 + 4f(t) + t + 2 = 0)$.

let $P(t)$ be any rational cubic interpolation function; if $P(t) \geqslant (\text{or} \leqslant) g(t)$ for all $t \in [t_0, t_n]$, then $P(t)$ is called the constrained interpolant above (or below) $g(t)$ on $[t_0, t_n]$. This interpolation is called constrained interpolation. In [5], the problem that constrains the rational interpolant curves defined by (1) to lie above a given piecewise linear curve was considered and the corresponding sufficient conditions were obtained. But there are no such positive parameters α_i and β_i to make the rational cubic spline curve defined by (1) to lie above or below the given piecewise lines in some cases. In the following, the condition for the weighted rational cubic spline curve $P(t)$ defined by (3) to lie above the straight line $g(t)$ or below the straight line $g^*(t)$ or between these two given straight lines in $[t_i, t_{i+1}]$ will be given, which show that the constrained problem could be solved completely by choosing the parameters α_i, β_i and the weighted coefficient λ. The condition of such constraint of the rational cubic curves defined by (1) given in [5] is the special case of $\lambda = 1$.

Theorem 1. *Given $\{(t_i, f_i, g_i, g_i^*, d_i), i=0,1,\dots,n\}$ with $g_i \leqslant f_i \leqslant g_i^*$, if the knots are equally spaced, then the sufficient condition for the weighted rational cubic spline curve $P(t)$ defined by (3) to lie above the straight line $g(t)$ and below the straight line $g^*(t)$ in $[t_i, t_{i+1}]$ is that the positive parameters α_i, β_i and $\lambda \in \mathbb{R}^+$ satisfy the following inequalities*:

$$\alpha_i[\lambda(f_i - f_{i+1} + h_i d_i) + (f_i + f_{i+1} - g_{i+1} - g_i)] + \beta_i(f_i - g_i) \geqslant 0, \tag{5}$$

$$\alpha_i(f_{i+1} - g_{i+1}) + \beta_i[\lambda(f_{i+2} - f_{i+1} - h_i d_{i+1}) + (3f_{i+1} - f_{i+2} - g_{i+1} - g_i)] \geqslant 0, \tag{6}$$

$$\alpha_i[\lambda(-f_i + f_{i+1} - h_i d_i) + (g_{i+1}^* + g_i^* - f_{i+1} - f_i)] + \beta_i(g_i^* - f_i) \geqslant 0, \tag{7}$$

$$\alpha_i(g_{i+1}^* - f_{i+1}) + \beta_i[\lambda(-f_{i+2} + f_{i+1} + h_i d_{i+1}) + (-3f_{i+1} + f_{i+2} + g_{i+1}^* + g_i^*)] \geqslant 0. \tag{8}$$

Proof Consider the case first that the weighted rational cubic spline curve $P(t)$ defined by (3) should lie below the straight line $g^*(t)$ in $[t_i, t_{i+1}]$. From (3), it is easy to know that for all $t \in [t_i, t_{i+1}]$, $q_i(t) \geqslant 0$. So,

$$P(t) = \frac{p_i(t)}{q_i(t)} \leqslant g^*(t)$$

is equivalent to

$$p_i(t) - q_i(t) g^*(t) \leqslant 0.$$

Let

$$U_i(t) = p_i(t) - q_i(t) g^*(t),$$

then

$$\begin{aligned} U_i(t) = {} & (1-\theta)^3 \alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta) W_i + \theta^3 \beta_i f_{i+1} \\ & -((1-\theta)\alpha_i + \theta\beta_i)((1-\theta) g_i^* + \theta g_{i+1}^*) \leqslant 0. \end{aligned} \tag{9}$$

Since

$$\begin{aligned} & ((1-\theta)\alpha_i + \theta\beta_i)((1-\theta) g_i^* + g_{i+1}^* \theta) \\ & \quad = (1-\theta)^2 \alpha_i g_i^* + \theta(1-\theta)(\alpha_i g_{i+1}^* + \beta_i g_i^*) + \theta^2 \beta_i g_{i+1}^* \\ & \quad = (1-\theta)^3 \alpha_i g_i^* + \theta(1-\theta)^2(\alpha_i g_{i+1}^* + \beta_i g_i^* + \alpha_i g_i^*) \\ & \qquad + \theta^2(1-\theta)(\alpha_i g_{i+1}^* + \beta_i g_i^* + \beta_i g_{i+1}^*) + \theta^3 \beta_i g_{i+1}^*, \end{aligned}$$

(9) becomes

$$U_i(t) = (1-\theta)^3 \alpha_i (f_i - g_i^*) + \theta(1-\theta)^2 A_i + \theta^2(1-\theta) B_i + \theta^3 \beta_i (f_{i+1} - g_{i+1}^*) \leqslant 0,$$

where

$$\begin{aligned} A_i &= V_i - (\alpha_i g_{i+1}^* + \beta_i g_i^* + \alpha_i g_i^*) \\ &= \lambda\alpha_i(f_i - f_{i+1} + h_i d_i) + \alpha_i(f_i + f_{i+1} - g_{i+1}^* - g_i^*) + \beta_i(f_i - g_i^*), \end{aligned}$$

$$B_i = W_i - (\alpha_i g^*_{i+1} + \beta_i g^*_i + \beta_i g^*_{i+1})$$
$$= \lambda\beta_i(f_{i+2} - f_{i+1} - h_i d_{i+1}) + \beta_i(3f_{i+1} - f_{i+2} - g^*_{i+1} - g^*_i) + \alpha_i(f_{i+1} - g^*_{i+1}).$$

Since

$$\alpha_i(f_i - g^*_i) \leqslant 0,$$
$$\beta_i(f_{i+1} - g^*_{i+1}) \leqslant 0$$

if $A_i \leqslant 0$, and $B_i \leqslant 0$, then $U_i(t) \leqslant 0$ holds for all $t \in [t_i, t_{i+1}]$.

In the same way, the sufficient condition for the interpolating curves to lie above a given straight line $g(t)$ in a given interval $[t_i, t_{i+1}]$ is as follows:

$$\lambda\alpha_i(f_i - f_{i+1} + h_i d_i) + \alpha_i(f_i + f_{i+1} - g_{i+1} - g_i) + \beta_i(f_i - g_i) \geqslant 0,$$
$$\lambda\beta_i(f_{i+2} - f_{i+1} - h_i d_{i+1}) + \beta_i(3f_{i+1} - f_{i+2} - g_{i+1} - g_i) + \alpha_i(f_{i+1} - g_{i+1}) \geqslant 0.$$

Thus, the proof is complete. □

Now the problem is whether the weight coefficient $\lambda \in \mathbb{R}$ exists to ensure the weighted rational cubic spline remains C^1-continuous in the whole interval $[t_0, t_n]$ and whether positive parameters α_i, β_i exist for each interval $[t_i, t_{i+1}]$ to make the weighted rational cubic spline curve lie above the straight line $g(t)$ (the "above case") and/or below the straight line $g^*(t)$ (the "below case") simultaneously. Consider the case $g_i < f_i < g^*_i$.

Theorem 2. *For the given data $\{(t_i, f_i, g_i, g^*_i, d_i),\ i = 0, 1, \ldots, n\}$ with $g_i < f_i < g^*_i$, there must exist $\lambda \in \mathbb{R}$ to ensure the weighted rational cubic spline remains C^1-continuous in the whole interval $[t_0, t_n]$ and positive parameters α_i, β_i exist for each interval $[t_i, t_{i+1}]$ to make the weighted rational cubic spline curve lie above the straight line $g(t)$ and below the straight line $g^*(t)$ simultaneously.*

Proof Let $\eta_i = \alpha_i/\beta_i$, then for the "Above Case", (5) and (6) can be rewritten as

$$\eta_i[\lambda(f_i - f_{i+1} + h_i d_i) + (f_i + f_{i+1} - g_{i+1} - g_i)] + (f_i - g_i) \geqslant 0, \tag{10}$$

$$\eta_i(f_{i+1} - g_{i+1}) + [\lambda(f_{i+2} - f_{i+1} - h_i d_{i+1}) + (3f_{i+1} - f_{i+2} - g_{i+1} - g_i)] \geqslant 0. \tag{11}$$

Since $f_i > g_i$, $i = 0, 1, \ldots, n$, if $f_i - f_{i+1} + h_i d_i \geqslant 0$, then for any $\lambda > 0$ and $\eta_i > 0$, (10) holds; if $f_i - f_{i+1} + h_i d_i < 0$, denote $\lambda_i = (f_i + f_{i+1} - g_{i+1} - g_i)/(f_{i+1} - f_i - h_i d_i)$, then $\lambda_i > 0$. Choose a λ such that $\lambda \leqslant \lambda_i$, then for any $\eta_i > 0$, (10) holds. For this λ, no matter what the expression $\lambda(f_{i+2} - f_{i+1} - h_i d_{i+1}) + (3f_{i+1} - f_{i+2} - g_{i+1} - g_i)$ is, there must exist a real number $M_i > 0$ such that when $\eta_i \geqslant M_i$ (11) holds. Thus, for the given data $\{(t_i, f_i, g_i, d_i),\ i = 0, 1, \ldots, n\}$, no matter what the expression $f_i - f_{i+1} + h_i d_i$ is, there exists a $\lambda_i > 0$ and a $M_i > 0$ such that, when $0 < \lambda < \lambda_i$ and $\eta_i > M_i$, then (10) and (11) hold.

In a similar way, for the "Below Case", (7) and (8) can be rewritten as

$$\eta_i[\lambda(-f_i + f_{i+1} - h_i d_i) + (g^*_{i+1} + g^*_i - f_{i+1} - f_i)] + (g^*_i - f_i) \geqslant 0, \tag{12}$$

$$\eta_i(g^*_{i+1} - f_{i+1}) + [\lambda(-f_{i+2} + f_{i+1} + h_i d_{i+1}) + (-3f_{i+1} + f_{i+2} + g^*_{i+1} + g^*_i)] \geqslant 0. \tag{13}$$

In a similar way to the "Above Case", for the given data $\{(t_i, f_i, g_i^*, d_i),\ i = 0, 1, \dots, n\}$, no matter what the real value of the expression $f_{i+1} - f_i - h_i d_i$ is, there exists a $\lambda_i^* > 0$ and $M_i^* > 0$ such that when $0 < \lambda < \lambda_i^*$ and $\eta_i > M_i^*$, then (12) and (13) hold.

Combining the discussions in the two cases, and letting

$$\lambda^* = \min\{\lambda_0, \lambda_1, \dots, \lambda_{n-1}, \lambda_0^*, \lambda_1^*, \dots, \lambda_{n-1}^*\},$$

$$M^* = \max\{M_0, M_1, \dots, M_{n-1}, M_0^*, M_1^*, \dots, M_{n-1}^*\},$$

choose $\lambda < \lambda^*$ and $\eta > M^*$, then the weighted rational cubic spline curves defined by (3) are not only C^1-continuous in the interval $[t_0, t_n]$, but are bounded between the two given piecewise lines $g(t)$ and $g^*(t)$.

The proof is complete. $\square$

3.1. A numerical example

Let $f(t) = \sin(\pi t/2), t \in [0, 4]$, and the knots $t_i = ih$ $(i = 0, 1, \dots, 8)$, $h = 0.5$. Denote the weighted rational cubic interpolating function $P(t)$ (of $f(t)$) by (3). Let the "above" constraining curve be $g^*(t)$ with

$$g^*(t) = \begin{cases} 0.07 + \frac{7}{5}t, & 0 \leqslant t \leqslant 0.5, \\ 0.47 + \frac{3}{5}t, & 0.5 \leqslant t \leqslant 1.0, \\ 1.67 - \frac{3}{5}t, & 1.0 \leqslant t \leqslant 1.5, \\ 2.87 - \frac{7}{5}t, & 1.5 \leqslant t \leqslant 2.5, \\ 0.87 - \frac{3}{5}t, & 2.5 \leqslant t \leqslant 3.0, \\ -2.73 + \frac{3}{5}t, & 3.0 \leqslant t \leqslant 3.5, \\ -5.53 + \frac{7}{5}t, & 3.5 \leqslant t \leqslant 4.0, \end{cases}$$

and let the "below" constraining curve be $g(t)$ with

$$g(t) = \begin{cases} -0.07 + \frac{7}{5}t, & 0 \leqslant t \leqslant 0.5, \\ 0.33 + \frac{3}{5}t, & 0.5 \leqslant t \leqslant 1.0, \\ 1.53 - \frac{3}{5}t, & 1.0 \leqslant t \leqslant 1.5, \\ 2.73 - \frac{7}{5}t, & 1.5 \leqslant t \leqslant 2.5, \\ 0.73 - \frac{3}{5}t, & 2.5 \leqslant t \leqslant 3.0, \\ -2.87 + \frac{3}{5}t, & 3.0 \leqslant t \leqslant 3.5, \\ -5.67 + \frac{7}{5}t, & 3.5 \leqslant t \leqslant 4.0. \end{cases}$$

Let $\lambda = 1.2$, $\eta_i = \alpha_i/\beta_i$, with $\eta_i = 0.1$, $i = 1, 2, 3, 5, 6, 7$; and $\eta_i = 0.3$ for $i = 4, 8$, then the interpolating weighted spline curves $P(t)$ defined by (3) are being constrained between two given piecewise lines $g^*(t)$ and $g(t)$. Fig. 7 shows the interpolating curve which is being constrained between two given curves $g^*(t)$ and $g(t)$. For the same function $f(t)$ being interpolated and the same constrained curves

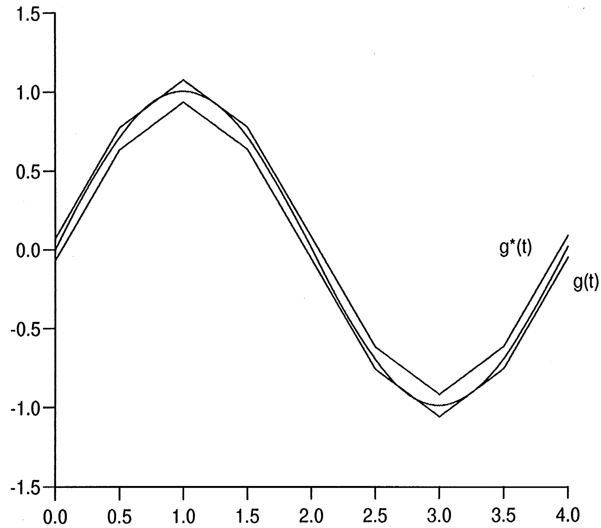

Fig. 7. $P(t)$ is being constrained between two given curves.

$g^*(t)$ and $g(t)$, there are no such parameters $\eta_i = \alpha_i/\beta_i$ which ensure that the rational spline curves $P^*(t)$ defined by (1) to be constrained between $g^*(t)$ and $g(t)$. If the same parameters $\eta_i = \alpha_i/\beta_i$, with $\eta_i = 0.1$, $i = 1,2,3,5,6,7$; and $\eta_i = 0.3$, $i = 4,8$ are used, it is found that the rational spline curves $P^*(t)$ cannot be bounded between the two given curves $g^*(t)$ and $g(t)$.

Table 1 gives the values of the functions $g^*(t)$, $P^*(t)$, $P(t)$, $g(t)$ and $f(t)$ in the interpolating interval $[0,2]$.

4. Approximation properties of the weighted interpolation

As Section 3 shows, the weighted interpolation defined by (3) can be relied upon to constrain the interpolatory spline curve to be above, below or between two given piecewise lines, more so than the interpolation defined by (1). How well the weighted interpolation function approximates the function being interpolated is of interest. The following is a study of the approximation properties of the weighted interpolation function.

Let $P(t)$ be the weighted interpolatory function defined by (3). Using the Peano–Kernel Theorem gives the following.

Theorem 3. *Let $f(t) \in C^2[a,b]$ and $P(t)$ be the weighted rational cubic interpolatory function of $f(t)$ in $[t_i, t_{i+1}]$ (Eq. (3)). When the knots are equally spaced, for the fixed parameters α_i, β_i, and the weighted coefficient λ,*

$$|R[f]| = |f(t) - P(t)| = \left| \int_{t_i}^{t_{i+2}} f^{(2)}(\tau) R_t[(t-\tau)_+]\,\mathrm{d}\tau \right|$$

Table 1
The values of $g^*(t)$, $P^*(t)$, $P(t)$, $g(t)$ and $f(t)$ in [0,2]

t	$g^*(t)$	$P^*(t)$	$P(t)$	$g(t)$	$f(t)$
0.0	0.07000	0.00000	0.00000	−0.07000	0.00000
0.1	0.21000	0.16234	0.15706	0.07000	0.15643
0.2	0.35000	0.31696	0.30650	0.21000	0.30902
0.3	0.49000	0.45958	0.44801	0.35000	0.45399
0.4	0.63000	0.58968	0.58155	0.49000	0.58779
0.5	0.77000	0.70711	0.70711	0.63000	0.70711
0.6	0.83000	0.81116	0.80686	0.69000	0.80902
0.7	0.89000	0.89361	0.88302	0.75000	0.89101
0.8	0.95000	0.95268	0.94029	0.81000	0.95106
0.9	1.01000	0.98817	0.97923	0.87000	0.98769
1.0	1.07000	1.00000	1.00000	0.93000	1.00000
1.1	1.01000	0.98481	0.98402	0.87000	0.98769
1.2	0.95000	0.94679	0.94229	0.81000	0.95106
1.3	0.89000	0.88771	0.88176	0.75000	0.89101
1.4	0.83000	0.80780	0.80329	0.69000	0.80902
1.5	0.77000	0.70711	0.70711	0.63000	0.70711
1.6	0.63000	0.58462	0.58841	0.49000	0.58779
1.7	0.49000	0.44853	0.45338	0.35000	0.45399
1.8	0.35000	0.30457	0.30892	0.21000	0.30902
1.9	0.21000	0.15477	0.15477	0.07000	0.15643
2.0	0.07000	0.00000	0.00000	−0.07000	0.00000

$$
\begin{aligned}
&\leqslant \|f^{(2)}(t)\| \left[\int_{t_i}^{t} |p(\tau)|\,\mathrm{d}\tau + \int_{t}^{t_{i+1}} |q(\tau)|\,\mathrm{d}\tau + \int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\,\mathrm{d}\tau \right] \\
&= \|f^{(2)}(t)\| W(\theta, \alpha_i, \beta_i, \lambda),
\end{aligned}
\tag{14}
$$

where

$$
R_t[(t-\tau)_+] = \begin{cases}
(t-\tau) - \frac{1}{(1-\theta)\alpha_i+\beta_i\theta}[\theta(1-\theta)^2\alpha_i(1-\lambda)(t_{i+1}-\tau) + \theta^2(1-\theta)(\alpha_i + 3\beta_i \\
\quad -\lambda\beta_i)(t_{i+1}-\tau) - \beta_i(1-\lambda_i)(t_{i+2}-\tau) - \lambda\beta_i h_i) + \beta_i\theta^3(t_{i+1}-\tau)], \\
\quad t_i < \tau < t, \\
-\frac{1}{(1-\theta)\alpha_i+\beta_i\theta}[\theta(1-\theta)^2\alpha_i(1-\lambda)(t_{i+1}-\tau) + \theta^2(1-\theta)(\alpha_i + 3\beta_i \\
\quad -\lambda\beta_i)(t_{i+1}-\tau) - \beta_i(1-\lambda)(t_{i+2}-\tau) - \lambda\beta_i h_i) + \beta_i\theta^3(t_{i+1}-\tau)], \\
\quad t < \tau < t_{i+1}, \\
\frac{1}{(1-\theta)\alpha_i+\beta_i\theta}\theta^2(1-\theta)(1-\lambda)\beta_i(t_{i+2}-\tau), \quad t_{i+1} < \tau < t_{i+2}.
\end{cases}
$$

$$
= \begin{cases} (t-\tau) - \frac{1}{(1-\theta)\alpha_i+\beta_i\theta}[\alpha_i\theta(1-\theta) - \lambda\alpha_i\theta(1-\theta)^2 + (2\theta^2-\theta^3)\beta_i)(t_{i+1}-\tau) \\ \quad -\theta^2(1-\theta)\beta_i h_i], \\ -\frac{1}{(1-\theta)\alpha_i+\beta_i\theta}[\alpha_i\theta(1-\theta) - \lambda\alpha_i\theta(1-\theta)^2 + (2\theta^2-\theta^3)\beta_i)(t_{i+1}-\tau) \\ -\theta^2(1-\theta)\beta_i h_i], \quad t<\tau<t_{i+1}, \\ \frac{1}{(1-\theta)\alpha_i+\beta_i\theta}\theta^2(1-\theta)(1-\lambda)\beta_i(t_{i+2}-\tau), \quad t_{i+1}<\tau<t_{i+2}. \end{cases}
$$

$$
= \begin{cases} p(\tau), \ t_i<\tau<t, \\ q(\tau), \ t<\tau<t_{i+1}, \\ r(\tau), \ t_{i+1}<\tau<t_{i+2}, \end{cases}
$$

and

$$
W(\theta,\alpha_i,\beta_i,\lambda) = \int_{t_i}^{t} |p(\tau)|\,\mathrm{d}\tau + \int_{t}^{t_{i+1}} |q(\tau)|\,\mathrm{d}\tau + \int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\,\mathrm{d}\tau. \tag{15}
$$

Eq. (14) may be written as

$$
\begin{aligned} \|R[f]\| = \|f(t)-P(t)\| &\leqslant \|f^{(2)}(t)\| W(\theta,\alpha_i,\beta_i,\lambda) \\ &\leqslant \|f^{(2)}(t)\| \max_{0\leqslant\theta\leqslant 1} W(\theta,\alpha_i,\beta_i,\lambda) \\ &= \|f^{(2)}(t)\| h_i^2 c_i(\eta_i,\lambda), \end{aligned} \tag{16}
$$

say, where $\eta_i = \alpha_i/\beta_i$.

To find $c_i(\eta_i,\lambda)$, the integrals on the left-hand side of Eq. (15) need to be calculated. Using Eq. (14), it can be shown that

$$
\int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\,\mathrm{d}\tau = \frac{|1-\lambda|\theta^2(1-\theta)}{2\{(1-\theta)\eta_i+\theta\}} h_i^2. \tag{17}
$$

To find $\int_{t_i}^{t} |p(\tau)|\,\mathrm{d}\tau$ and $\int_{t}^{t_{i+1}} |q(\tau)|\,\mathrm{d}\tau$, three cases need to be defined.

Case 1: $\lambda\leqslant 0$

For this case, it can be shown that

$$
\int_{t_i}^{t} |p(\tau)|\,\mathrm{d}\tau = \frac{\theta^2(1-\theta)^2\{\eta_i - \lambda(2-\theta)\eta_i + \theta\}}{2\{(1-\theta)\eta_i+\theta\}} h_i^2 \tag{18}
$$

and

$$
\begin{aligned} &\int_{t}^{t_{i+1}} |q(\tau)|\,\mathrm{d}\tau \\ &= \frac{\theta(1-\theta)^2[(1-\theta)^2\{1-\lambda(1-\theta)\}^2\eta_i^2 + 2\theta(1-\theta)^2\{1-\lambda(1-\theta)\}\eta_i + \theta^2(\theta^2-2\theta+2)]}{2\{(1-\theta)\eta_i+\theta\}\{(1-\theta)\eta_i - \lambda(1-\theta)^2\eta_i + \theta(2-\theta)\}} h_i^2. \end{aligned} \tag{19}
$$

The function $c_i(\eta_i,\lambda)$ is thus given by

$$
c_i(\eta_i,\lambda) = \max_{0\leqslant\theta\leqslant 1} h_i^{-2}\left[\int_{t_i}^{t} |p(\tau)|\,\mathrm{d}\tau + \int_{t}^{t_{i+1}} |q(\tau)|\,\mathrm{d}\tau + \int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\,\mathrm{d}\tau\right], \tag{20}
$$

where the integrals in Eq. (20) are given by (18), (19) and (17), respectively.

Case 2: $0 < \lambda \leqslant 1$

For this case, it can be shown that $\int_t^{t_{i+1}} |q(\tau)|\,\mathrm{d}\tau$ is given by Eq. (19), while

$$\begin{aligned}\int_{t_i}^{t} |p(\tau)|\,\mathrm{d}\tau = {} & \frac{\theta^2(1-\theta)^2}{2\{(1-\theta)\eta_i+\theta\}\{\eta_i+\lambda\theta\eta_i+\theta\}^2} \\ & \times[\{1-(2-3\theta)\lambda+(3\theta^2-4\theta+2)\lambda^2+\theta(\theta^2-2\theta+2)\lambda^3\}\eta_i^3 \\ & +\theta\{3-2(2-3\theta)\lambda+(3\theta^2-4\theta+2)\lambda^2\}\eta_i^2+\theta^2\{3-(2-3\theta)\lambda\}\eta_i+\theta^3]h_i^2.\end{aligned} \tag{21}$$

The function $c_i(\eta_i, \lambda)$ is given by (20) with the three integrals given now by (21), (19) and (17), respectively.

Case 3: $\lambda > 1$

Let $\theta_1 = (\lambda - 1)/(\lambda + (1/\eta_i))$. If $0 \leqslant \theta \leqslant \theta_1$, then

$$\begin{aligned}c_i(\eta_i,\lambda) &= \max_{0\leqslant\theta\leqslant\theta_1} h_i^{-2}\left[\int_{t_i}^{t}|p(\tau)|\,\mathrm{d}\tau+\int_{t}^{t_{i+1}}|q(\tau)|\,\mathrm{d}\tau+\int_{t_{i+1}}^{t_{i+2}}|r(\tau)|\,\mathrm{d}\tau\right] \\ &= \frac{1}{2}(\lambda-1)\max_{0\leqslant\theta\leqslant\theta_1}\theta(1-\theta),\end{aligned} \tag{22}$$

where

$$\int_{t_i}^{t} |p(\tau)|\,\mathrm{d}\tau = -\frac{\theta^2(1-\theta)^2\{\eta_i-\lambda(2-\theta)\eta_i+\theta\}}{2\{(1-\theta)\eta_i+\theta\}}h_i^2 \tag{23}$$

and

$$\int_{t_i}^{t} |q(\tau)|\,\mathrm{d}\tau = -\frac{\theta(1-\theta)^2\{(1-\theta)\eta_i-\lambda(1-\theta)^2\eta_i-\theta^2\}}{2\{(1-\theta)\eta_i+\theta\}}h_i^2. \tag{24}$$

If $\theta_1 \leqslant \theta \leqslant 1$, then $c_i(\eta_i, \lambda)$ is the same as in case 2. As there are two values of the error constants c_i in this case, a minimum value of c_i (Eq. (22) and case 2) is chosen.

In Fig. 8, a two-dimensional plot of $c_i(\eta_i, \lambda)$ is given. From this figure, it is found that for a large value of λ (in modulus) the error constant c_i is quite large, whereas for a small value of λ ($-0.5 \leqslant \lambda \leqslant 2$), the error constant is small. The maximum values of $c_i(\eta_i, \lambda)$ for the cases 1, 2 and 3 are found using the NAG library package E04ABF (for maximizing a function).

In Fig. 9, a plot of the error constants c_i for some fixed values of λ ($\lambda = 0, 0.5, 1, 1.1$ and 3) is given. From this figure, it can be seen that the error constants $c_i(\eta_i, \lambda)$ are smaller for $\lambda = 1$. It can also be seen, however, that the results obtained for $c_i(\eta_i, \lambda)$ for values of λ around 1 (for example, $\lambda = 0.8, 1.1, \ldots$, see Fig. 9) are quite similar to those given for $\lambda = 1$.

Acknowledgements

The authors are very grateful to the referees for their valuable comments and suggestions which have improved the paper. The support of the Royal Society of London and the Natural Science Foundation of China is gratefully acknowledged.

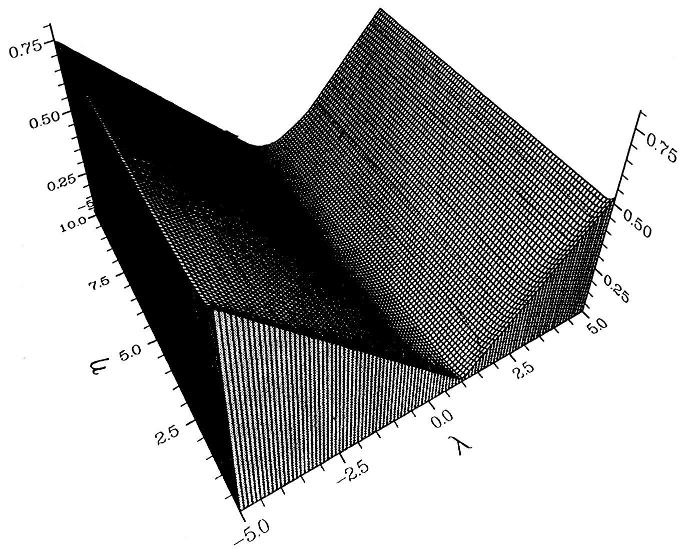

Fig. 8. The error constant $c_i(\eta_i, \lambda)$.

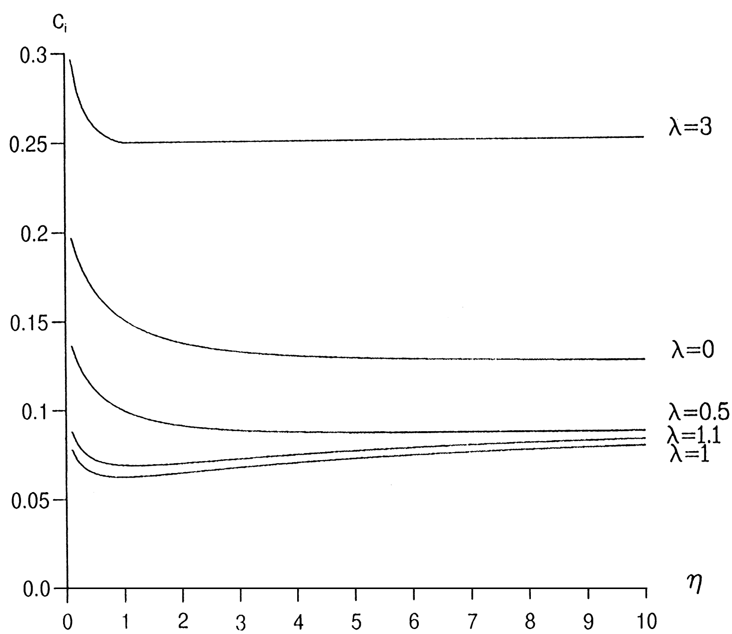

Fig. 9. The error constant $c_i(\eta_i, \lambda)$ for $\lambda = 0, 0.5, 1, 1.1$ and 3.

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